Week 08: System Analysis in the Time Domain

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Lecture Overview

- General Description
- First-order and Second-order Systems

System Analysis

- Mathematical Modeling of Physical Systems
- Transfer function and characteristic polynomial
- Typical Test Signals: Impulse, step, and ramp functions
- Analytical solution and computer simulation methods

- Natural and Forced Responses

- **Transient** and Steady-State Response

Input-Output Model

Linear, time-invariant system that is initially at rest

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y^{(1)} + a_0 y$$

$$= b_m u^{(m)} + b_{m-1}u^{(m-1)} + \dots + b_1 u^{(1)} + b_0 u$$

The transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \qquad m \le n$$

causality

- **Poles** are the roots of the denominator polynomial
- **Zeros** are the roots of the numerator polynomial
- Poles and zeros are either real numbers or they appear as complex conjugates.

Order of the system

- The degree of the polynomial in the denominator of the transfer function

$$s^{n} + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

- The number of poles of the transfer function

$$G(s) = \frac{b_m(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

- Minimum number of first order differential equations
- Number of state variables

First Order Systems

- Systems with a transfer function that has a characteristic polynomial of degree one

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

The pole of the system is at $p = -\frac{1}{\tau}$

Time constant

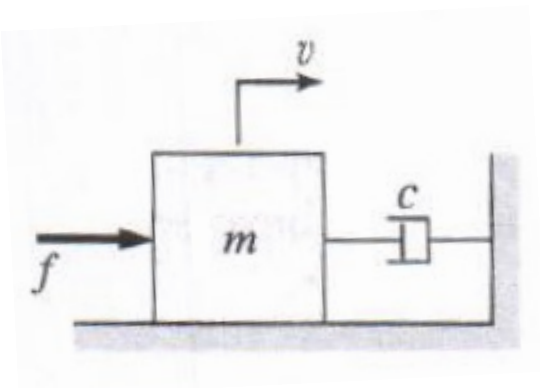
<u>Steady State Gain</u>

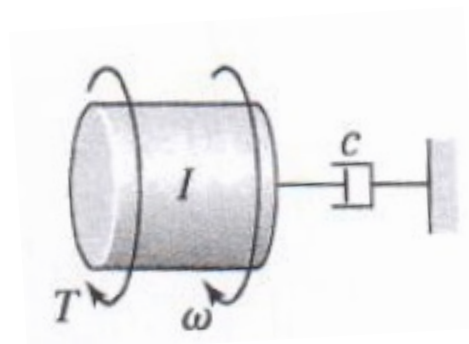
$$K = \lim_{s \to 0} G(s)$$

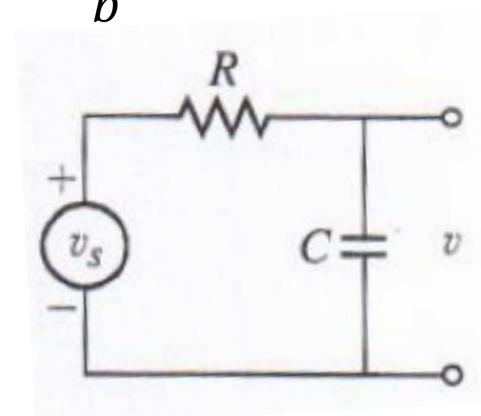
First Order Physical Systems

- General form:

$$a\frac{dy(t)}{dt} + by(t) = u(t) \qquad G(s) = \frac{Y(s)}{U(s)} = \frac{1/b}{\frac{a}{b}s + 1}$$







$$m\frac{dv(t)}{dt} + cv(t) = f(t)$$

$$\tau = \frac{m}{c}$$

$$I\frac{d\omega(t)}{dt} + c\omega(t) = T(t)$$

$$\tau = \frac{I}{c}$$

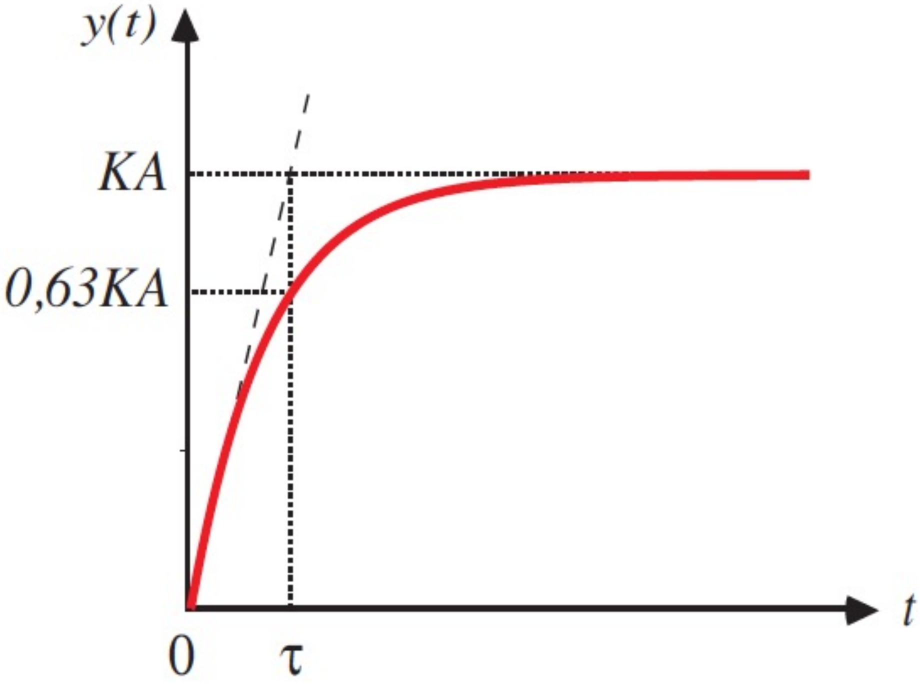
$$RC\frac{dv(t)}{dt} + v(t) = v_s(t)$$

$$\tau = RC$$

First Order Systems: Step Response

$$u(t) = A\varepsilon(t)$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{K}{(\tau s + 1)}\frac{A}{s}\right] = \varepsilon(t)KA[1 - e^{-t/\tau}]$$



$$t = \tau$$

$$y(\tau) = KA\left(1 - \frac{1}{e}\right) = 0,63KA$$

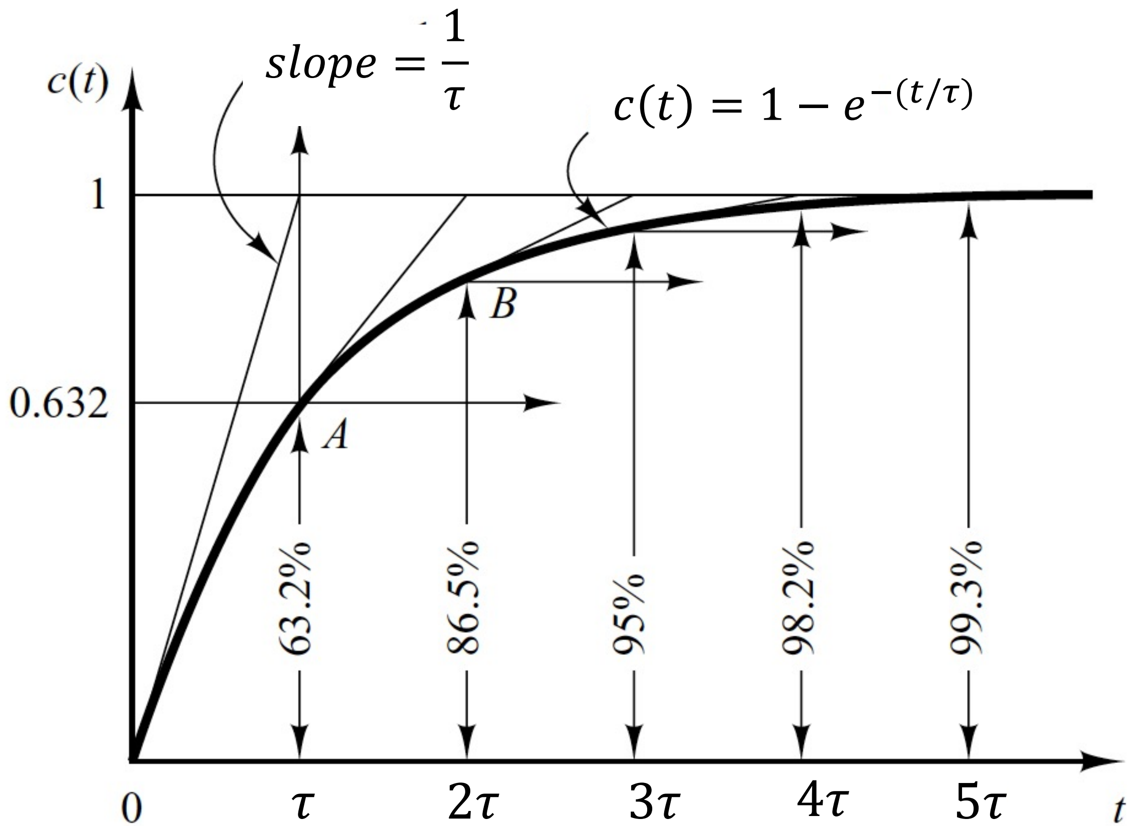
$$t = 3\tau$$

$$y(3\tau) = KA\left(1 - \frac{1}{e^3}\right) = 0,95KA$$

First Order Systems: Step Response

$$c(t) = \frac{y(t)}{KA}$$

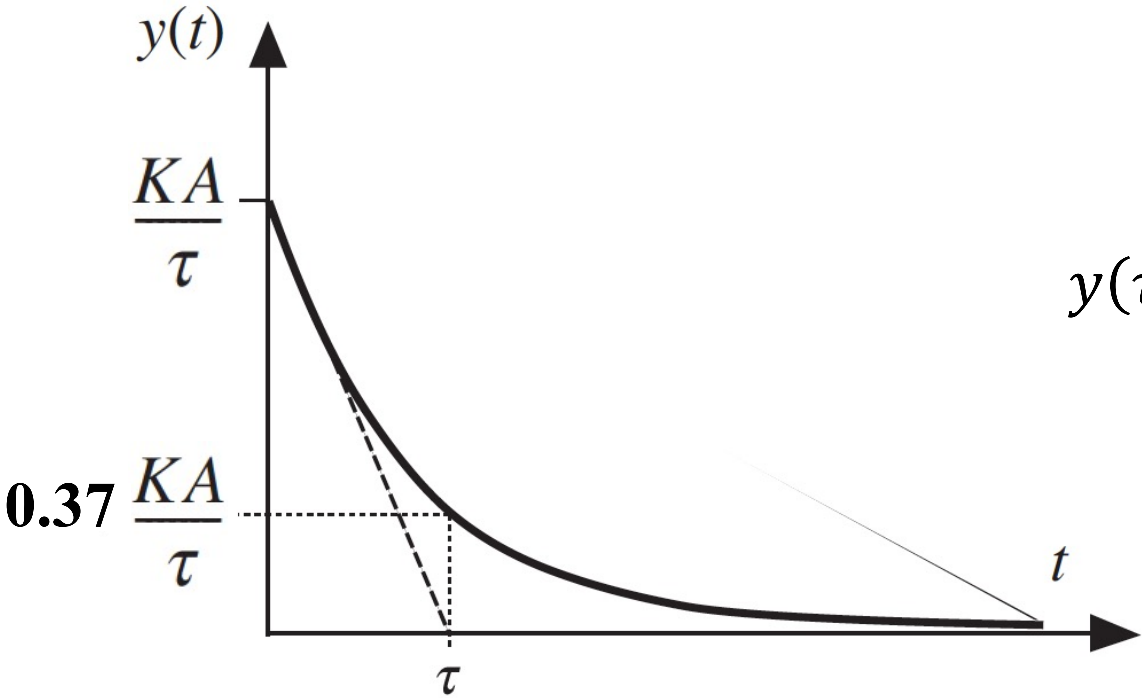
Normalized response curve



First Order Systems: Impulse Response

$$u(t) = A\delta(t)$$

$$Y(s) = \frac{KA}{\tau s + 1} \xrightarrow{\mathcal{L}^{-1}} y(t) = \frac{KA}{\tau} e^{-t/\tau}$$



$$t = \tau$$

$$y(\tau) = \frac{KA}{\tau e} = 0.37\frac{KA}{\tau}$$

First Order Systems: Ramp Response

$$u(t) = At$$

$$Y(s) = \left(\frac{K}{\tau s + 1}\right)\frac{1}{s^2} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{\tau s + 1}$$

$$A_1 = \lim_{s\to 0}\{Y(s)s^2\} = \lim_{s\to 0}\left\{\frac{KA}{\tau s + 1}\right\} = KA$$

$$A_2 = \lim_{s\to 0}\left\{\frac{d}{ds}[Y(s)s^2]\right\} = \lim_{s\to 0}\left\{\frac{-KA\tau}{(\tau s + 1)^2}\right\} = -KA\tau$$

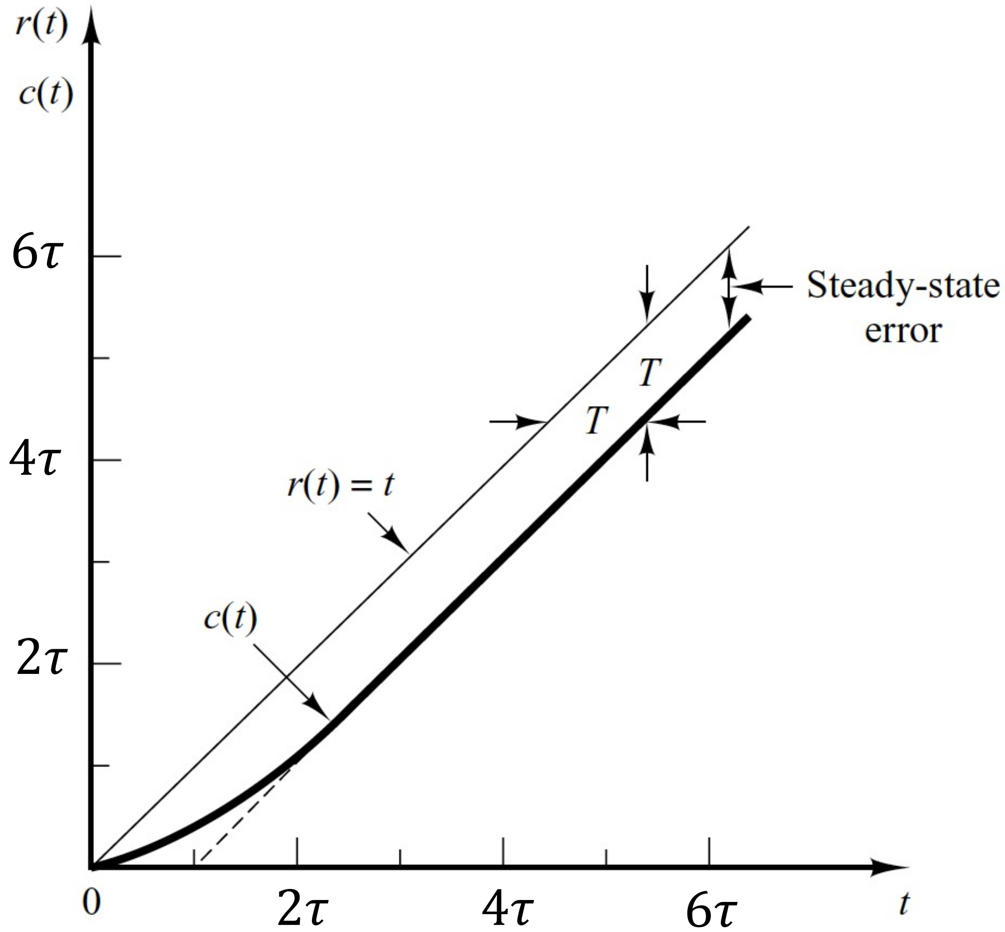
$$A_3 = \lim_{s\to -1/\tau}\{Y(s)(\tau s + 1)\} = \lim_{s\to -1/\tau}\left\{\frac{KA}{s^2}\right\} = KA\tau^2$$

$$y(t) = -KA\tau + KAt + KA\tau e^{-t/\tau} = KA(t - \tau) + KA\tau e^{-t/\tau}$$

First Order Systems: Ramp Response

$$c(t) = \frac{y(t)}{KA}$$

Normalized response curve



A property of LTI Systems

- (Normalized) Ramp Response

$$c(t) = t - \tau + \tau e^{-(t/\tau)} \qquad for\ t \geq 0$$

- (Normalized) Step Response is the derivative of the (normalized) ramp response

$$c(t) = 1 - e^{-(t/\tau)} \qquad for\ t \geq 0$$

- (Normalized) Impulse Response is the derivative of the (normalized) step response

$$c(t) = \frac{1}{\tau} e^{-(t/\tau)} \qquad for\ t \geq 0$$

Example

- Impulse response of a system is given by

$$g(t) = 3e^{-0.5t}$$

- Find the time constant, DC gain, and unit step response

$$G(s) = \frac{3}{s+0.5} = \frac{6}{2s+1}$$

$$\tau = 2 \text{ and } K = 6$$

$$Y(s) = \frac{1}{s} \times \frac{3}{s+0.5} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$y(t) = 6 - 6e^{-0.5t}$$

System Identification: Experimental Protocol

- Apply unit step input
- Measure steady-state value of the output (gain)
- Measure the time to reach a percentage of the gain (time constant)

Second Order Systems

- Transfer function without zeros

$$G(s) = \frac{Y(s)}{U(s)} = K \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

K : Steady-state output (DC Gain)
ζ : Damping ratio
ω_0 : (Undamped) Natural frequency

- For now, assume that $\zeta \geq 0$

Second Order Systems

- Transfer function without zeros

$$G(s) = \frac{Y(s)}{U(s)} = K \frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

- **Poles of the System**

$$p_{1,2} = -\omega_0 \left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

- Poles are either
  - distinct real number,
  - repeated real numbers, or
  - complex conjugates

Second Order Systems

- Overdamped response (real and distinct poles)
- Critically damped response (real and repeated poles)
- Underdamped response (complex conjugate poles)
- Un-damped response (complex conjugate poles without real parts)

Partial Fraction Expansion

- **CASE 1:** Distinct real roots

$$Y(s) = \frac{N(s)}{(s + r_1)(s + r_2) \cdots (s + r_n)}$$

- **CASE 2:** Distinct complex roots

$$Y(s) = \frac{s + 1}{s(s^2 + 4s + 5)}$$

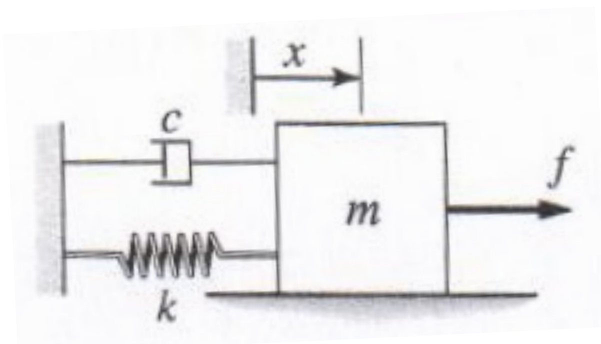
- **CASE 3:** Repeated real roots

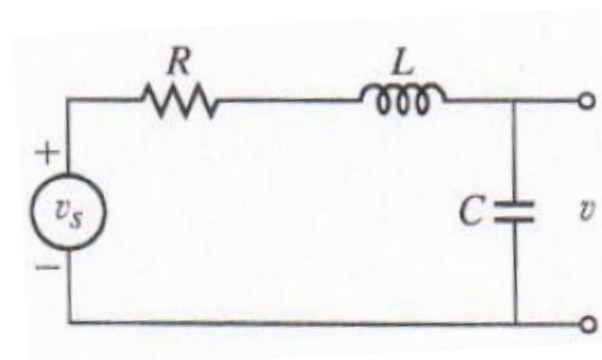
$$Y(s) = \frac{N(s)}{(s + r_1)^p \left(s + r_{p+1}\right) \cdots (s + r_n)}$$

Second Order Physical Systems

- General form:

$$a\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = u(t) \qquad G(s) = \frac{1/a}{s^2 + \frac{b}{a}s + \frac{c}{a}}$$





$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = f(t)$$

$$LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) = v_s(t)$$

Second Order Systems

- **Overdamped Case (Poles are real and distinct)** $\zeta > 1$

$$G(s) = K \frac{1}{(\tau_1 s + 1)} \frac{1}{(\tau_2 s + 1)}$$

$$\tau_{1,2} = \frac{1}{\omega_0\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)}$$

- **Step Response**

$$u(t) = A\varepsilon(t)$$

$$y(t) = \varepsilon(t)KA\left\{1 - \frac{1}{\tau_1 - \tau_2}[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}]\right\}$$

Second Order Systems

- **Step Response**

$$u(t) = A\varepsilon(t)$$

$$y(t) = \varepsilon(t)KA\left\{1 - \frac{1}{\tau_1 - \tau_2}\left[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right]\right\}$$

- **Example**

$$u(t) = \varepsilon(t) \qquad G(s) = \frac{1}{(4s+1)(s+1)} = \frac{0.25}{s^2 + 1.25s + 0.25}$$

$$y(t) = 1 - \frac{1}{3}\left[4e^{-t/4} - e^{-t}\right]$$

$$G(s) = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2}$$

$$\tau_{1,2} = \frac{1}{\omega_0\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)} = -\frac{1}{p_{1,2}}$$

Special Case: Dominant Root Approximation

When $\zeta \gg 1$

- One of the two decaying exponentials decreases much faster than the other
- Faster decaying exponential term may be neglected (smaller time constant)
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system.
- Dominant pole is the one closest to the origin

$$y(t) = A_1 e^{-3t} + A_1 e^{-15t}$$

Dominant term

Second Order Systems

- **Critically damped Case (Poles are real and equal)** $\zeta = 1$

$$G(s) = \frac{Y(s)}{U(s)} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = \frac{K}{(\tau s + 1)^2}$$

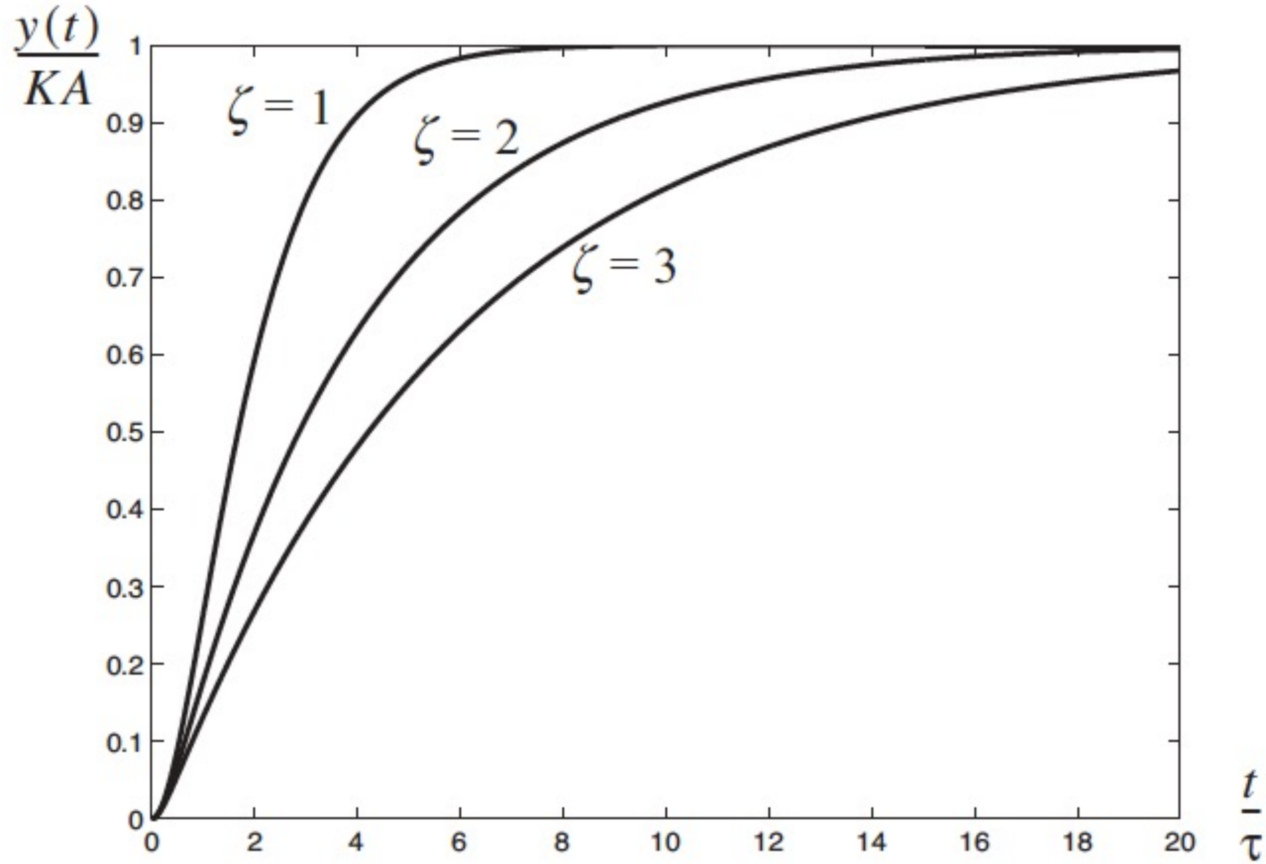
$$\tau = \frac{1}{\omega_0\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)} = \frac{1}{\omega_0}$$

- **Step Response**

$$u(t) = A\varepsilon(t)$$

$$y(t) = \varepsilon(t)KA\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right] = \varepsilon(t)KA[1 - e^{-\omega_0 t}(1 + \omega_0 t)]$$

Step Response of Second Order Systems



Second Order Systems

- **Underdamped Case (Poles are complex conjugates)** $0 \leq \zeta < 1$

$$G(s) = \frac{Y(s)}{U(s)} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = K\frac{a^2 + \bar{\omega}^2}{(s+a)^2 + \bar{\omega}^2}$$

Damped natural frequency

$$\bar{\omega} = \omega_0\sqrt{1-\zeta^2}$$

Attenuation

$$a = \zeta\omega_0$$

Second Order Systems

- **Underdamped Case (Poles are complex conjugates)** $0 \leq \zeta < 1$

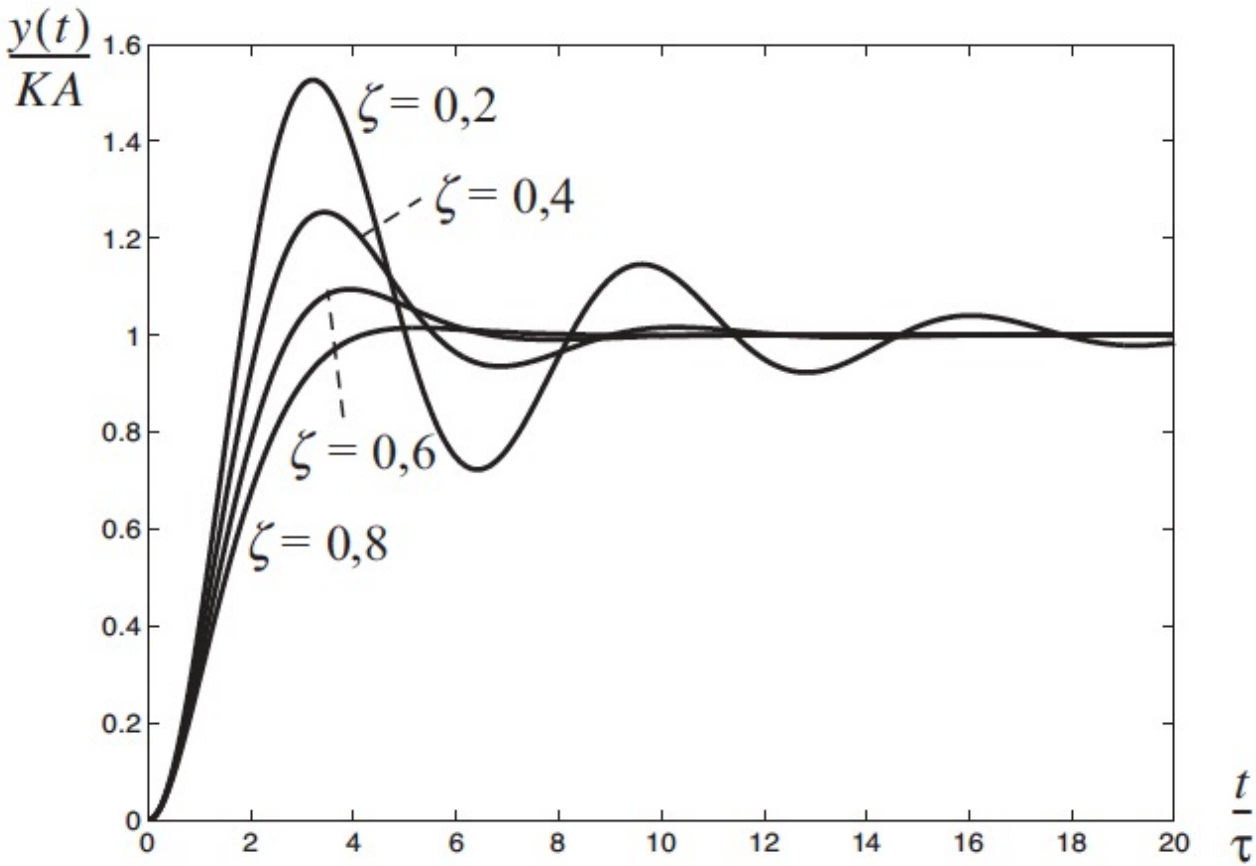
$$G(s) = \frac{Y(s)}{U(s)} = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = K\frac{a^2 + \overline{\omega}^2}{(s+a)^2 + \overline{\omega}^2}$$

- **Step Response**

$$u(t) = A\varepsilon(t) \qquad Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_0}{(s+\zeta\omega_0)^2 + \overline{\omega}^2} - \frac{\zeta\omega_0}{(s+\zeta\omega_0)^2 + \overline{\omega}^2}$$

$$y(t) = \varepsilon(t)KA\left\{1 - e^{-at}\left(\cos\overline{\omega}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\overline{\omega}t\right)\right\}$$

Step Response of Second Order Systems



Special Case: Undamped Response

When $\zeta = 0$

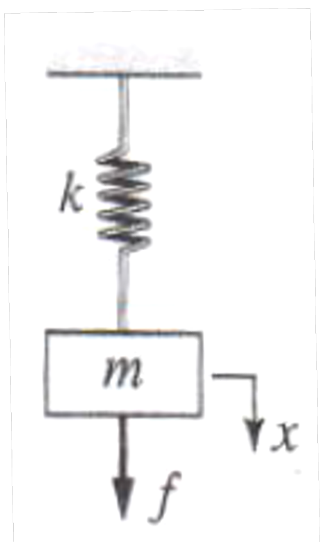
- The response becomes undamped
- Oscillations continue indefinitely with natural frequency

$$y(t) = \varepsilon(t)KA[1 - \cos \omega_0 t]$$

Second Order Physical Systems

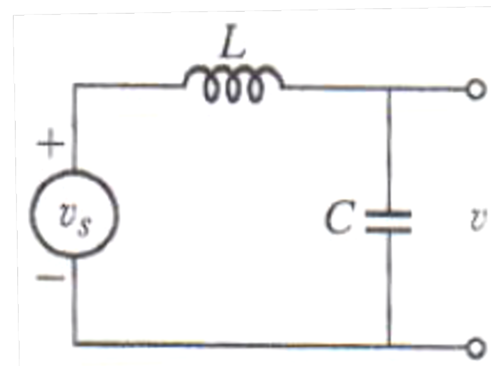
- General Form:

$$a\frac{d^2y(t)}{dt^2} + cy(t) = u(t) \qquad G(s) = \frac{1/a}{s^2 + \frac{c}{a}}$$



$$m\frac{d^2x(t)}{dt^2} + kx(t) = f(t)$$

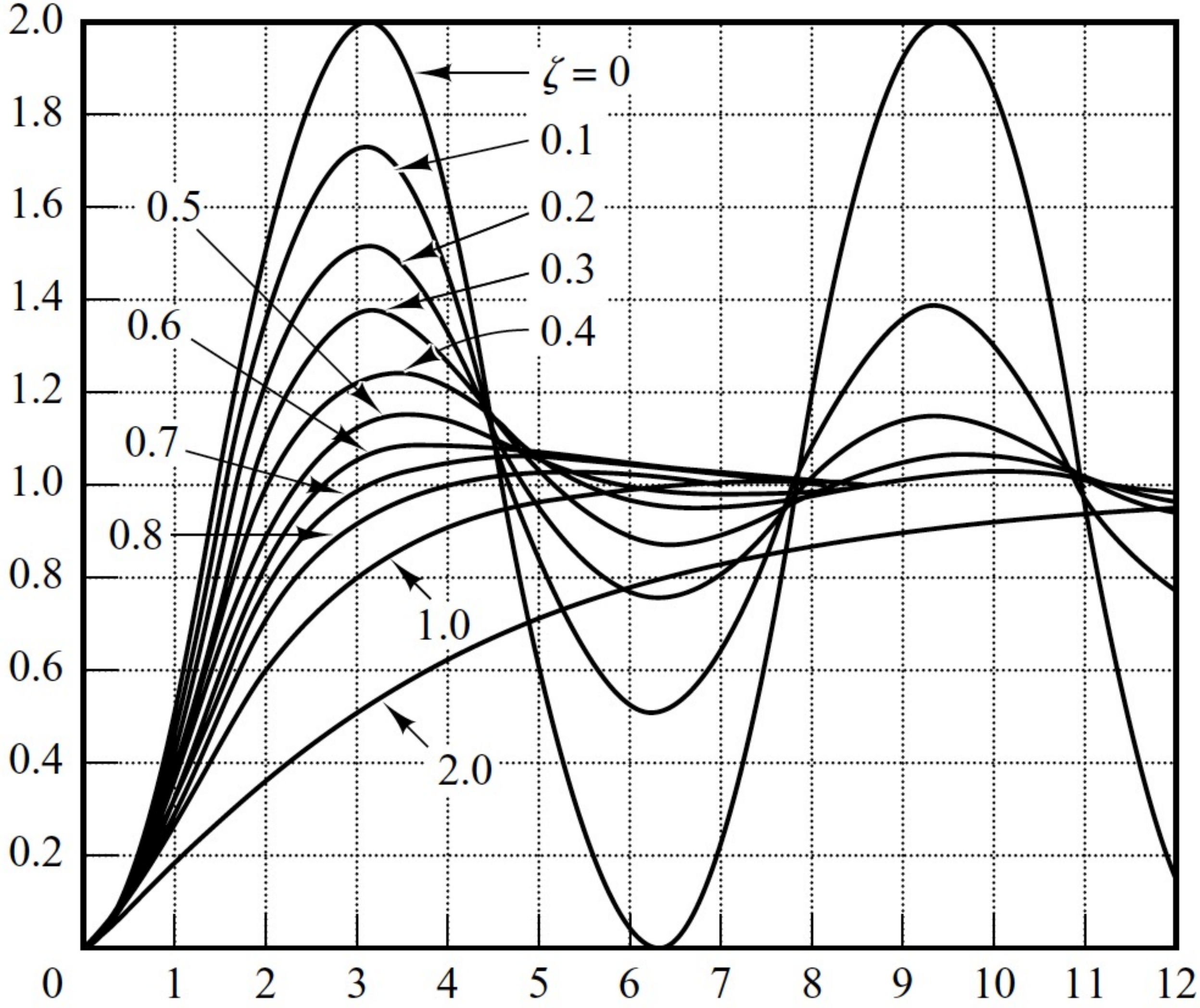
$$\omega_0 = \sqrt{\frac{k}{m}}$$



$$LC\frac{d^2v(t)}{dt^2} + v(t) = v_s(t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Step Response of Second Order Systems



Transient Response: Underdamped

- **Delay time, t_d:** Time required for the response to reach half of the final value the very first time

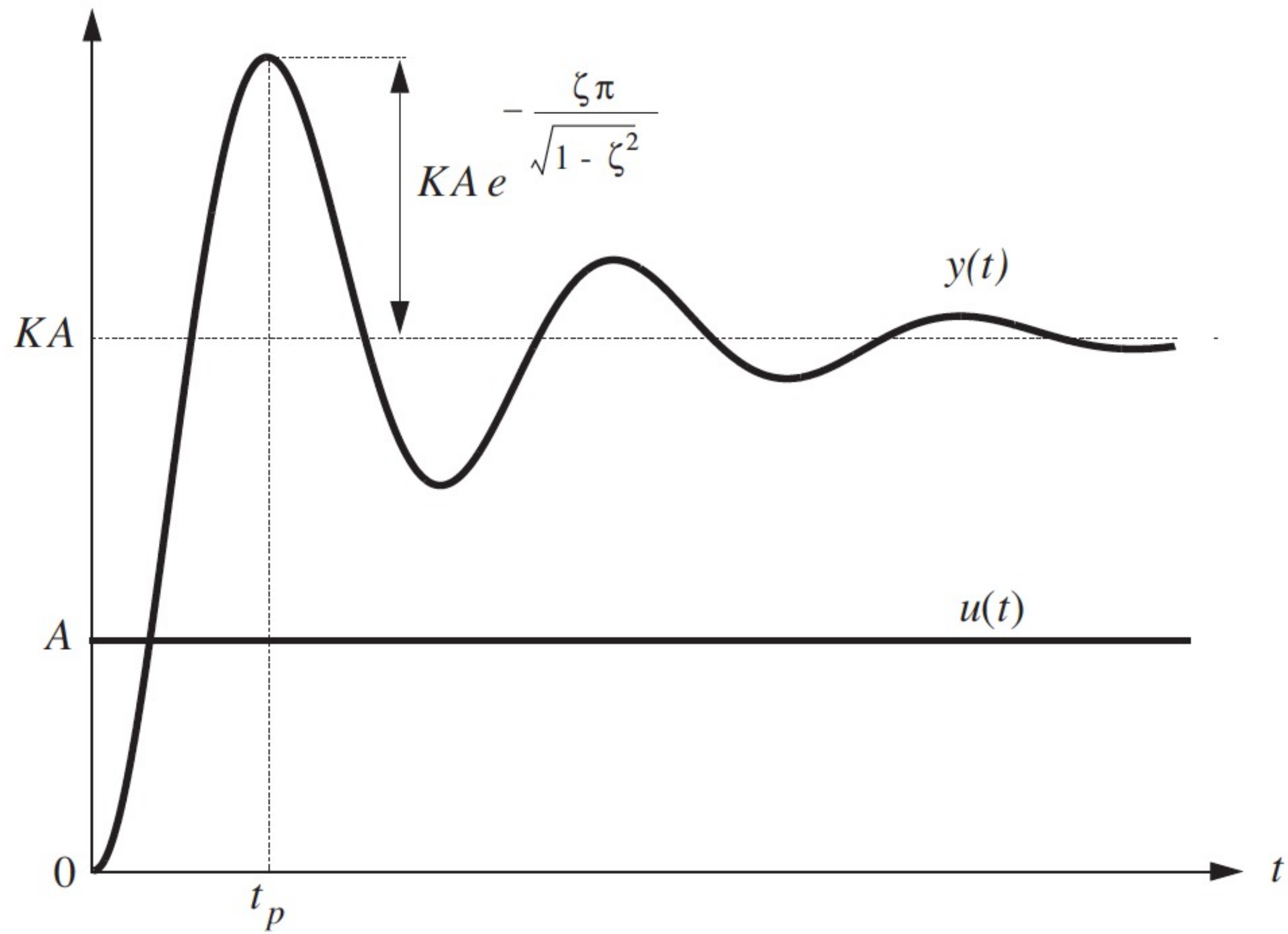
- **Rise time, t_t:** Time required for the response to rise from 0% to 100% (underdamped system) or from 10% to 90% (overdamped system)

- **Peak time, t_p:** Time required for the response to reach the first peak of the overshoot.

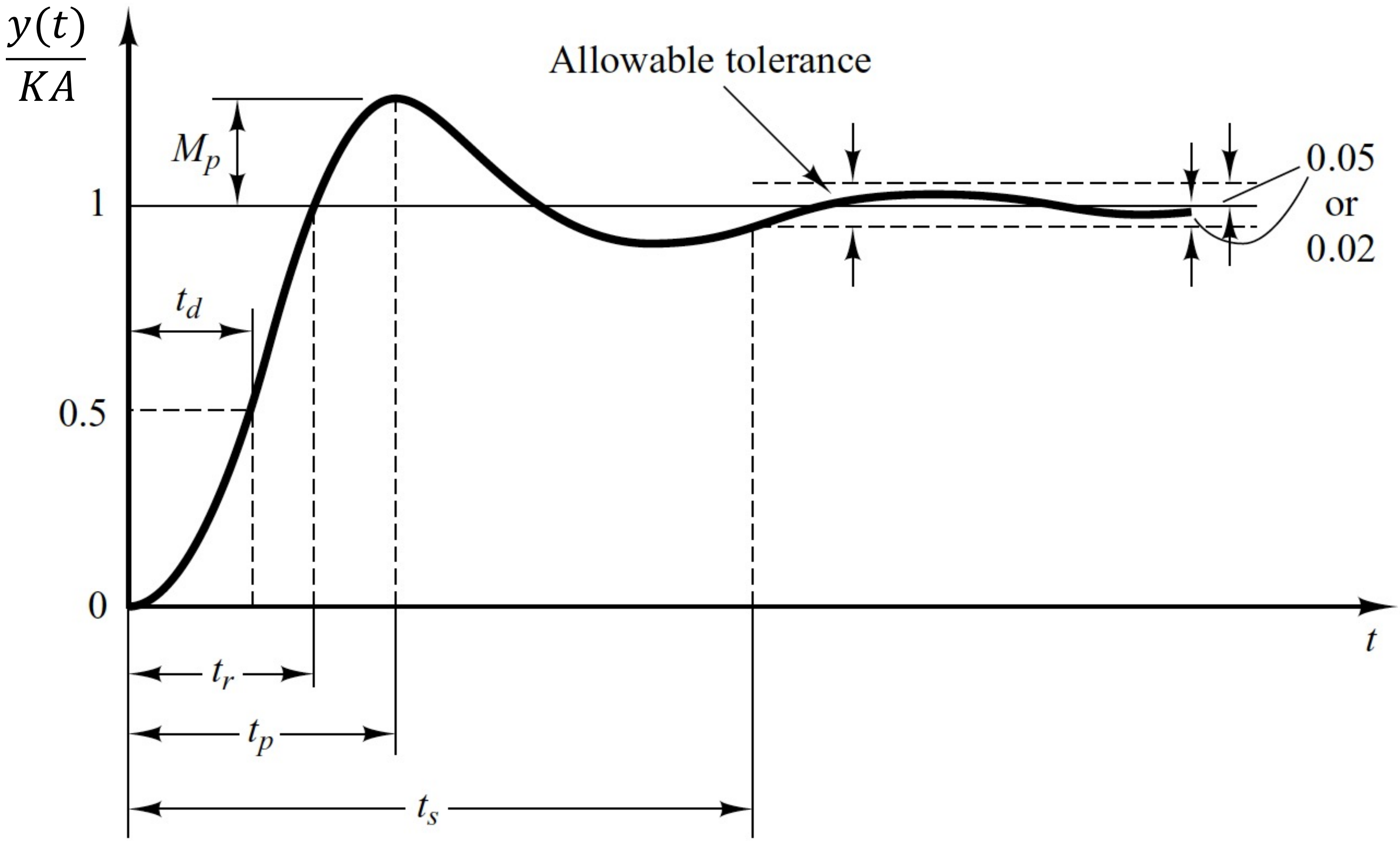
- **Maximum percent overshoot, M_p**

- **Settling time, t_s:** Time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%)

Transient Response: Underdamped



Transient Response: Underdamped

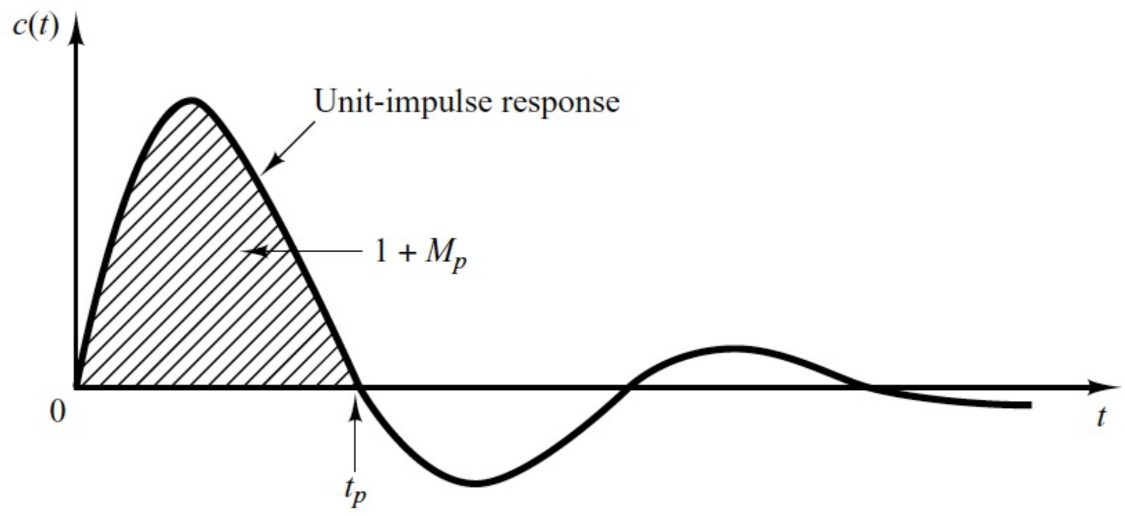


Impulse Response

$$u(t) = A\delta(t) \qquad G(s) = K\frac{{\omega_0}^2}{s^2 + 2\zeta\omega_0 s + {\omega_0}^2} = K\frac{a^2 + \bar{\omega}^2}{(s+a)^2 + \bar{\omega}^2}$$

For $0 \leq \zeta < 1$

$$y(t) = \varepsilon(t) KA \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-at} \sin \bar{\omega} t$$



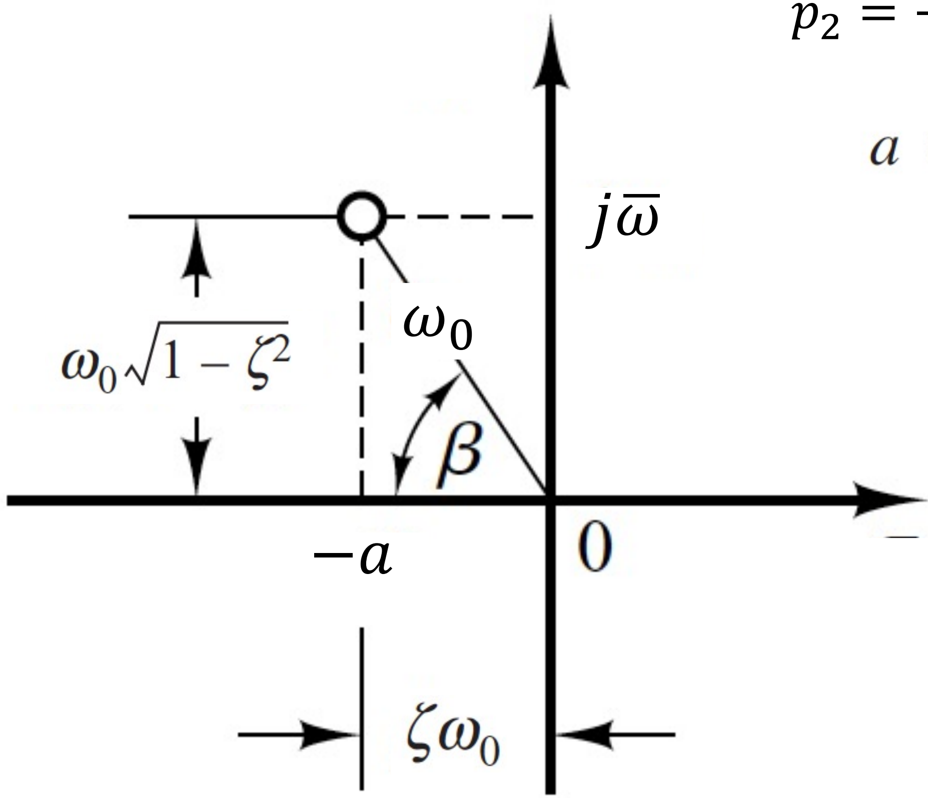
Graphical Representation

$$p_{1,2} = -\omega_0(\zeta \pm j\sqrt{1-\zeta^2})$$

$$p_1 = -\omega_0\zeta - j\omega_0\sqrt{1-\zeta^2}) = -a - j\bar{\omega}$$

$$p_2 = -\omega_0\zeta + j\omega_0\sqrt{1-\zeta^2}) = -a + j\bar{\omega}$$

$$a = \zeta\omega_0 \qquad \bar{\omega} = \omega_0\sqrt{1-\zeta^2}$$



$$t_r = \frac{\pi - \beta}{\bar{\omega}}$$

$$t_p = \frac{\pi}{\bar{\omega}}$$

$$t_s = \frac{3}{a}$$ 5% criterion

System Identification: Experimental Protocol

- Measure steady-state value of the output (gain)
- Measure the peak value of the output (damping coefficient)
- Measure the time to reach the peak value (natural frequency and time constant)

$$\tau = \frac{1}{a}$$